Differential Microphones Arrays based on Differential Equation

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***Abstract*—This report explains the uniformity of linear DMA (LDMA) and circular DMA (CDMA) based on differential equations. The beampatterns of circular and linear Differential Microphone Arrays are polynomial about sinusoidal function. Classical method of designing beampattern is solving simultaneous equations formed from constrains on the nulls. In this paper, we prove that the polynomial about sinusoidal function is the solution of a differential equation and the differential equation corresponding to LDMA and CDMA are same. When changing the initial values of the differential equation properly, the solution of the differential equation can be beampatterns of LDMA or beampatterns of CDMA. Therefore, one differential equation can represent two kinds of geometry arrays.**

***Index Terms*—Differential microphone array, array geometry, differential equation**

I. INTRODUCTION

DIFFERENTIAL microphone arrays (DMA) are often used to extract a desired source signal from noise. Beampattern differential microphones can date back to more than 50 years ago. Compared with other arrays, DMA have many advantages. For example, the directivity of DMA is high, which makes it widespread used in human-machine interface systems [1]. The design of an array geometry is an important part. Linear and circular DMA are two common type of geometry. The selection of the geometry depends heavily on the application requirements. Linear geometry is preferable as this type of arrays can be easily integrated into smartphones and PCs. In applications like teleconferencing and 3D sound recording where the signal of interest many come from any direction, it is necessary for the microphone array to have similar, if not the same, response from one direction to another. In this case, circular arrays are often used. Another important advantage of using circular arrays is that the processing problem can be greatly simplified due to the symmetry property [2].

Recently, a new approach to the design and implementation of uniform linear DMAs was developed [3]. This method works in the short-time Fourier transform (STFT) domain as illustrated in Fig. 1. Based on the new method of beamforming, the conventional method to calculate the expressions of Nth-order beampatterns is that we get N+1 linear equations from the constrains on N nulls and solve the equations. In this paper, we use differential equation to calculate the expression of beampatterns. The organization of this paper is as fellow. In section II, we discuss the linear DMA's beampattern based on linear equations and differential equations. In section III, we illustrate circular DMA's beampattern based on differential equations. In section IV, we summarize the similarity between two kinds of geometry.

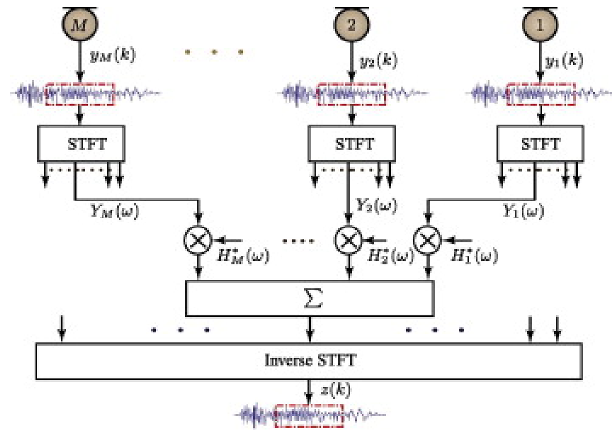

Fig. 1. A schematic diagram of a DMA system in the STFT domain.

II. LINEAR DMA

A. Signal model

The basic mode is uniform linear array (ULA) with M omnidirectional microphones as illustrated in Fig. 2 [3].

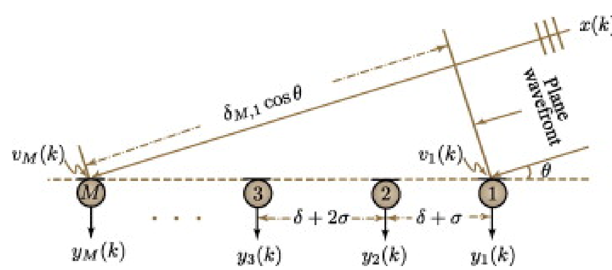

Fig. 2. Illustration of a uniform linear microphone array.

With the assumption that the source is in the far field, the signal received at mth microphone and at time k can be written as

$$y_m(k) = x_m(k) + v_m(k) \tag{1}$$

$$= x_m(k - t - \tau_m) + v_m(k), m = 1, 2, ..., M \tag{2}$$

where $x_m(k)$ is the source signal, t is the time which it takes form the signal to the first microphone, τ_m is the delay between the mth and the first microphones. As for ULA structure,

$$\tau_m = \frac{(m-1)\delta\cos\theta}{c} \tag{3}$$

where δ is the spacing between adjacent microphones, θ is the source incidence angle, and c is the speed of sound in the air.

In the STFT domain,(1) can be expressed as

$$Y_m(\omega) = X(\omega)e^{-j(m-1)\omega\tau_0\cos\theta} + V_m(\omega) \tag{4}$$

where $\tau_0 = \frac{\delta}{c}$ is the relative delay between adjacent microphones at the angle $\theta = 0°$, $\omega = 2\pi f$ is the angular frequency, $j = \sqrt{-1}$ is the imaginary unit and $Y_m(\omega)$, $X(\omega)$, $V_m(\omega)$ are the frequency-domain representations of $y_m(k)$, $x(k)$, $v_m(k)$, respectively. Putting all these signals in a vector form, we get

$$y(\omega) = [Y_1(\omega), Y_2(\omega), ..., Y_M(\omega)]^T \tag{5}$$
$$= d(\omega, \cos\theta)\mathrm{X}(\omega) + v(\omega) \tag{6}$$

where

$$d(\omega, \cos\theta) = \left[1, e^{-j\omega\tau_0\cos\theta}, ..., e^{-j(M-1)\omega\tau_0\cos\theta}\right]^T \tag{7}$$

is the phase-delay vector of length M.

In order to recover the desired signal $X(\omega)$ from $y(\omega)$, a complex weight $H_m^*(\omega)$ is designed and applied to the output of each microphone, where the superscript $*$ denotes complex conjugation. All the weighted outputs are then summed together to produce an estimate of the clean signal as illustrated in Fig. 1. Mathematically, the beamformer's output is

$$Z(\omega) = \sum_{m=1}^{M} H_m^*(\omega)Y_{\mathrm{m}}(\omega) = h^T(\omega)y(\omega) \tag{8}$$

where where $Z(\omega)$ is supposed to be an estimate of $X(\omega)$, the superscript H is the conjugate–transpose operator, and

$$\mathrm{h}(\omega) = [H_1(\omega), H_2(\omega), ..., H_m(\omega)]^T \tag{9}$$

B. Beampatterns

The beampattern (or directivity pattern) describes the sensitivity of the beamformer to a plane wave (source signal) impinging on the array from the direction . Mathematically, it is written as θ. Mathematically, beampattern of a Nth-order DMA is written as

$$B[h(\omega), \theta] = d^H(\omega, \cos\theta)h(\omega) \tag{10}$$
$$= \sum_{m=1}^{M} H_m(\omega)e^{j(m-1)\omega\tau_0\cos\theta} \tag{11}$$

McLaughlin expand and simplify (11), we get

$$B_N(\theta) = \sum_{n=0}^{N} a_{N,n}\cos^n\theta \tag{12}$$

where $\mathrm{a}_{N,n}$,$n = 0, 1, ..., N$ are real coefficients. In the direction of the desired signal, i.e., $\theta = 0°$, the directivity pattern must be equal to 1. Therefore, we should have

$$\sum_{n=0}^{N} a_{N,n} = 1 \tag{13}$$

As a result, the first coefficient is determined,

$$a_{N,0} = 1 - \sum_{n=0}^{N} a_{N,n} \tag{14}$$

C. Linear equations solves beampattern coefficients

One of the critical issue in DMA beamforming is designing the geometry of beampattern by choosing proper $\mathrm{a}_{N,n}$. By (12), the Nth-order DMA has N null points and by (14), the number free coefficients of the Nth-order DMA is also N. Therefore, beampattern coefficients are determined by constraining the nulls, and then we can geometry the corresponding shape of pattern.

For example, the first-order directivity patterns have the form:

$$B_1(\theta) = a_{1,0} + a_{1,1}\cos\theta \tag{15}$$
$$= (1 - a_{1,1}) + a_{1,1}\cos\theta \tag{16}$$

They have 1 null at the angle θ_1, so we can write differential equation about $\mathrm{a}_{1,1}$,

$$(1 - a_{1,1}) + a_{1,1}\cos\theta_1 = 0 \tag{17}$$

By solving the equation, the most important shapes of patterns are as follows

- Dipole: $a_{1,1} = 1$, null at $\cos\theta = 0$.
- Cardioid: $a_{1,1} = \frac{1}{2}$, null at $\cos\theta = -1$.
- Hypercardioid: $a_{1,1} = \frac{2}{3}$, null at $\cos\theta = -\frac{1}{2}$.

The second-order directivity patterns have the form:

$$B_2(\theta) = (1 - a_{2,1} - a_{2,2}) + a_{2,1}\cos\theta + a_{2,2}\cos^2\theta \tag{18}$$

and they have 2 nulls at the angle θ_1 and θ_2, so we can write differential equation about $\mathrm{a}_{2,1}$ and $\mathrm{a}_{2,2}$,

$$\begin{cases} (1 - a_{2,1} - a_{2,2}) + a_{2,1}\cos\theta_1 + a_{2,2}\cos^2\theta_1 = 0 \\ (1 - a_{2,1} - a_{2,2}) + a_{2,1}\cos\theta_2 + a_{2,2}\cos^2\theta_2 = 0 \end{cases} \tag{19}$$

By solving the equation, the most important shapes of patterns are as follows

- Dipole: $a_{2,1} = 0$, $a_{2,2} = 1$, nulls at $\cos\theta = 0$.
- Cardioid: $a_{2,1} = a_{2,2} = \frac{1}{2}$, nulls at $\cos\theta = 0$ and $\cos\theta = -1$.
- Hypercardioid: $a_{2,1} = a_{2,2} = 1$.

As for Nth-order directivity patterns, they have N nulls at θ_1, θ_2,...,θ_N and the directivity pattern is equal to 1 in the direction of the desired signal. Based on the known conditions, we get

$$\begin{cases} \sum_{n=0}^{N} a_{N,n} = 1 \\ \sum_{n=0}^{N} a_{N,n}\cos^n\theta_1 = 0 \\ \sum_{n=0}^{N} a_{N,n}\cos^n\theta_2 = 0 \\ \vdots \\ \sum_{n=0}^{N} a_{N,n}\cos^n\theta_N = 0 \end{cases} \tag{20}$$

Solve the simultaneous linear equations and get $a_{N,0}$, $a_{N,2}$,...,$a_{N,N}$.

D. Differential equations solves beampattern coefficients

By (12) and the multiple-angle formula

$$\cos^n\theta = \frac{1}{2^n}\sum_{k=0}^{n} C_n^k \cos(2k-n)\theta \tag{21}$$

we get,

$$B_N(\theta) = \sum_{n=0}^{N} b_{N,n}\cos n\theta \tag{22}$$

Take the first-order equations as example, it is written as

$$B_1(\theta){=}b_{1,0} + b_{1,1}\cos\theta \tag{23}$$

As for a second-order constant coefficient differential equation

$$y^{(2)} + n^2 y = 0 \tag{24}$$

its corresponding characteristic equation is

$$r^2 + n^2 = 0 \tag{25}$$

so $r = \pm ni$, and its general solution is

$$y = C_1\cos(n\theta) + C_2\sin(n\theta) + C \tag{26}$$

where C is a constance. When $C_2 = 0$, $C_1 = b_{1,1}$ and $C = b_{1,0}$, solution of this differential equation is equal to (23). We give 3 specific initial conditions, and get (23) based on differential equation (24).

Therefore, characteristic equation of a Nth-order constant coefficient differential equation corresponding to Nth-order DMA is

$$r(r^2+1^2)(r^2+2^2)\dots(r^2+N^2) = 0 \tag{27}$$

To solve $(2N+1)$th-order differential equation needs 2N+1 initial conditions. By (14),

$$B_N(0^{\circ}) = 1 \tag{28}$$

Nth-order directivity patterns have N nulls at θ_1, θ_2,...,θ_N,

$$B_N(\theta_1) = 0, B_N(\theta_2) = 0, \dots, B_N(\theta_N) = 0 \tag{29}$$

Besides, (22) only contains constance and $\cos(n\theta)$, so its first derivative only exists $\sin(n\theta)$ and so on we can get the rest N initial condition,

$$B_N^{(1)}(0) = 0, B_N^{(1)}(\pi) = 0, B_N^{(2)}(\frac{\pi}{2}) = 0, B_N^{(2)}(\frac{3\pi}{2}) = 0... \tag{30}$$

III. Circular DMA

A. Signal Model

We consider a source signal (plane wave), in the farfield, that propagates in an anechoic acoustic environment at the speed of sound and impinges on a uniform circular array (UCA), of radius r, consisting of M omnidirectional microphones. The direction of the source signal to the array is parameterized by the azimuth angle θ. As illustrated in Fig. 3, when operating in the farfield, the time delay between microphone m and the center of the array is given by

$$\tau_m = \frac{r}{c}\cos(\theta - \varphi_m), m = 1, 2, ..., M \tag{31}$$

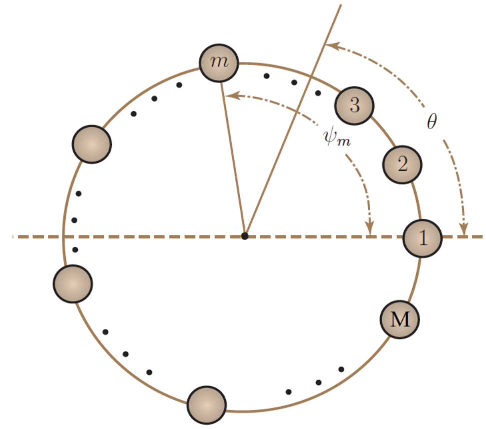

Fig. 3. Illustration of a uniform circular array with M microphones in the Cartesian coordinate system.

where

$$\varphi_m = \frac{2\pi(m-1)}{M} \tag{32}$$

is the angular position of the mth array element. In this scenario, the steering vector of length M is

$$d(\omega,\theta) = \begin{bmatrix} e^{j\omega r\cos(\theta-\varphi_1)/c} & \dots & e^{j\omega r\cos(\theta-\varphi_M)/c}\end{bmatrix}^T \tag{33}$$

where $\tau_0 = \frac{\delta}{c}$ is the relative delay between adjacent microphones at the angle $\theta = 0°$, $\omega = 2\pi f$ is the angular frequency, $j = \sqrt{-1}$ is the imaginary unit. For a UCA, the interelement spacing is

$$\delta = 2r\sin(\frac{\pi}{M}) \tag{34}$$

$$\approx \frac{2\pi r}{M} \tag{35}$$

Substituting the approximation of (35) into (33), the steering vector can be rewritten as

$$d(\omega,\theta) = \begin{bmatrix} e^{j\omega\delta\cos(\theta-\varphi_1)/c} & \dots & e^{jM\omega\delta\cos(\theta-\varphi_M)/c}\end{bmatrix}^T \tag{36}$$

Let us denote by θ_s the steering angle of the array. We consider designing fixed directional beamformers, like in DMAs, where the main lobe points at $\theta = \theta_s$ and the desired signal propagates from the same angle. For linear DMAs the optimal position is at $\theta = 0$ and electronic steering (in the sense that the main lobe can be oriented to any possible direction without affecting the shape of the beampattern) is not really feasible.We will see that with CDMAs, we have much more flexibilities [2].

B. Beampattern

The beampattern or directivity pattern describes the sensitivity of the beamformer to a plane wave (source signal) impinging on the UCA from the direction . Mathematically, it is defined as

$$B[h(\omega),\theta] = d^H(\omega,\theta_s)h(\omega) \tag{37}$$

$$= \sum_{m=1}^{M} H_m(\omega,\theta_s)e^{j\omega\cos(\theta-\varphi_m)} \tag{38}$$

where the superscript H is the conjugate-transpose operator. For any steering angle, θ_s, this beampattern is defined as

$$B_N(\theta) = \sum_{n=0}^{N} a_{N,n} \cos^n(\theta - \theta_s) \tag{39}$$

where $a_{N,n}$,$n = 0, 1, ..., N$ are real coefficients. In the direction of the desired signal, i.e., $\theta = \theta_s$, the directivity pattern must be equal to 1. Therefore, we should have

$$\sum_{n=0}^{N} a_{N,n} = 1 \tag{40}$$

C. Differential Equations Solve Circular DMA

By (39) and the multiple-angle formula

$$B_N(\theta - \theta_s) = \sum_{n=0}^{N} b_{N,n} \cos n(\theta - \theta_s) \tag{41}$$

$$= \sum_{n=0}^{N} b_{N,n} \cos n\theta_s \cos n\theta + b_{N,n} \sin n\theta_s \sin n\theta \tag{42}$$

Take the first-order equations as example, it is writtens as

$$B_1(\theta - \theta_s) = b_{1,0} + b_{1,1} \cos\theta_s \cos\theta + b_{1,1} \sin\theta_s \sin\theta \tag{43}$$

Based on Section 2.3, (26) is the solution of (24). When $b_{1,0} = C$, $C_1 = b_{1,1} \cos\theta_s$, $C_2 = b_{1,1} \sin\theta_s$, the solution of this differential equation is equal to (43). Therefore, characteristic equation of Nth-order circular DMA is also

$$r(r^2 + 1^2)(r^2 + 2^2) \ldots (r^2 + n^2) = 0 \tag{44}$$

To solve $(2N + 1)$th-order differential equation needs 2N+1 initial conditions. By (40)

$$B_N(\theta_s) = 1 \tag{45}$$

Nth-order directivity patterns have N nulls at $\theta_1 - \theta_s$, $\theta_2 - \theta_s$,...,$\theta_N - \theta_s$,

$$B_N(\theta_1 - \theta_s) = 0, B_N(\theta_2 - \theta_s) = 0, \ldots, B_N(\theta_N - \theta_s) = 0 \tag{46}$$

Besides, we can get the rest N initial conditions,

$$\begin{aligned} &B_N^{(1)}(\theta_s) = 0, B_N^{(1)}(\theta_s + \pi) = 0, \\ &B_N^{(2)}(\theta_s + \frac{\pi}{2}) = 0, B_N^{(2)}(\theta_s + \frac{3\pi}{2}) = 0... \end{aligned} \tag{47}$$

IV. Conclusion

From section 2 and 3, we prove the uniformity between linear DMA and circular DMA. As for a uniform Nth-order DMA, its corresponding differential equation is determined. Based on

$$r(r^2 + 1^2)(r^2 + 2^2) \ldots (r^2 + n^2) = 0, \tag{48}$$

we can get the corresponding $(2N + 1)$th-order differential equation. By the constrains on N nulls and beampattern's derivatives are equal to 0, we get 2N+1 initial value conditions,

$$\begin{cases} B_N(\theta_s - \theta_s) = 1 \\ B_N(\theta_1 - \theta_s) = 0 \\ \vdots \\ B_N(\theta_N - \theta_s) = 0 \\ B_N^{(1)}(\theta_s) = 0 \\ B_N^{(1)}(\theta_s + \pi) = 0 \\ B_N^{(1)}(\theta_s) = 0 \\ B_N^{(2)}(\theta_s + \frac{\pi}{2}) = 0 \\ B_N^{(2)}(\theta_s + \frac{3\pi}{2}) = 0 \\ \vdots \end{cases} \tag{49}$$

Take a second-order DMA as example, two nulls at the angle $\frac{\pi}{2}$ and π. $B_2(\theta)$'s characteristic equation is

$$r(r^2 + 1^2)(r^2 + 2^2) = 0 \tag{50}$$

so its differential equation is

$$y^{(5)} + 5y^{(3)} + 4y^{(1)} = 0 \tag{51}$$

As for a linear DMA,

$$\theta_s = 0 \tag{52}$$

and the initial values of the differential equation are

$$y(0) = 1, y(\frac{\pi}{2}) = 0, y(\pi) = 0, y^{(1)}(0) = 0, y^{(1)}(\pi) = 0 \tag{53}$$

Solve (51), we get

$$y = \frac{1}{4}\cos(2\theta) + \frac{1}{2}\cos(\theta) + \frac{1}{4} \tag{54}$$

Base on (54), we can draw the beampatter of second-order linear DMA as illustrated in Fig. 4

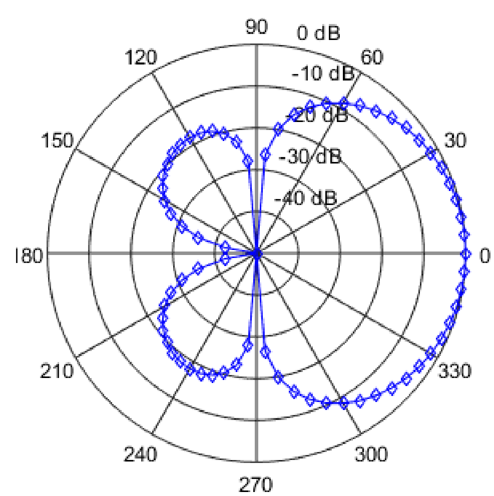

Fig. 4. The beampatter of second-order linear DMA.

As for a circular DMA, we set

$$\theta_s = \frac{\pi}{6} \tag{55}$$

and the initial values of the differential equation are

$$y(0-\frac{\pi}{6})=1, y(\frac{\pi}{2}-\frac{\pi}{6})=0, y(\pi-\frac{\pi}{6})=0,$$
$$y^{(1)}(0-\frac{\pi}{6})=0, y^{(1)}(\pi-\frac{\pi}{6})=0 \tag{56}$$

Solve (51), we get

$$y=\frac{1}{8}*\cos(2\theta)-\frac{1}{4}*\sin(\theta)-\frac{\sqrt{3}}{8}*\sin(2\theta)+\frac{\sqrt{3}}{4}*\cos(\theta)+\frac{1}{4} \tag{57}$$

Base on (57), we can draw the beampatter of second-order linear DMA as illustrated in Fig. 5 Compare Fig. 4 and Fig. 5,

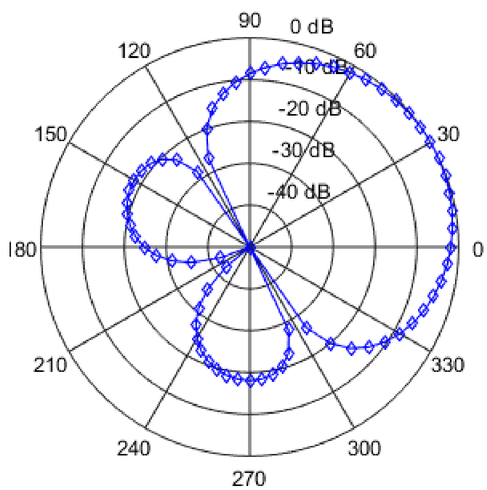

Fig. 5. The beampatter of second-order circular DMA.

we find that Fig. 4 rotating $\theta_s=\frac{\pi}{6}$ can get Fig. 5.

The circular DMA has the same differential equation as the linear DMA, and the difference between two kinds of DMA is initial value conditions. When $\theta_s=0$, the solution of the differential equation is linear DMA beampattern. To some extend, linear DMA is a special case of circular DMA.

Therefore, beampattern based on differential equations illustrates the uniformity of the same order DMA. As for a DMA with same order, whether it is circular or linear, its corresponding differential equation is equal. We only need to change initial value conditions of the differential equation and get different kinds of geometry of DMA.

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